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Graphical method for computing the determinant and inverse of a matrix: II. Generating functions for the $[a_n\ b_n\ 0\ 0\ \ldots]$ representation matrices of SU(n)

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Abstract. We consider a special $2n \times 2n$ matrix $1-x$ for which the determinant is the square of a polynomial in the x_{ij}. A graph G with n branches is associated to the matrix and $\det(1-x)$ and $(1-x)^{-1}_{ij}$ are expressed in terms of sums over subgraphs of G. The generating functions of the representation matrices $\mathscr{D}^{[m]}_{MM'}(u)$ of the representations $[m] = [a_n\ b_n\ 0\ \ldots]$ of SU(n) are expressed in terms of the determinant and inverse of such a $2n \times 2n$ matrix.

1. Introduction

In Labarthe 1978 (to be referred as I) we gave a graphical transcription of the usual algebraic expressions for the determinant and inverse of a matrix. The determinant was expressed as $\Sigma\, M(D)$ where the sum is over some special subgraphs (called closed diagrams) of the graph associated to the matrix and $M(D)$ is a monomial in the matrix elements. These results were similar to those of the flow graph theory of linear equations (Mason 1953, 1956, Coates 1959). In Labarthe (1975, to be referred as H) we also expressed graphically the determinant and inverse of a special matrix, obtaining expressions in terms of closed and open diagrams but quite different from the results of I. For example, the determinant was expressed as the *square* of the sum of $M(D)$ over the closed diagrams.

In § 2 we generalise the results of H, to a larger class of $2n \times 2n$ matrices, giving also a simpler proof for the inverse of the matrix. In § 3 we use the elegant formalism of Henrich (1975) describing representations of the unitary groups in entire function spaces with an Hermitian product given by a Gaussian measure. Generating functions for the representation matrices $\mathscr{D}^{[m]}_{MM'}(u)$ can be obtained as an integral from the generating functions of the basis vectors. This method has already been employed by Hassan (1979) who has entirely worked out the SU(3) case. These generating functions provide us with explicit expressions of the representation matrices. For a review of other methods giving explicit expressions see Louck and Biedenharn (1973). We consider here the case of the representations $[a_n\ b_n\ 0\ \ldots]$ of SU(n) which correspond to Young diagrams of not more than two rows. Generating functions for the basis vectors were obtained by Henrich (1975). The resulting generating functions for the $\mathscr{D}^{[m]}$ matrices are expressed in terms of the inverse and determinant of a $2n \times 2n$ matrix of the type considered in § 2.

2. Graphical method for computing $\det(1-x)$ and $(1-x)^{-1}$

We consider a $2n\times 2n$ complex matrix

$$x=\begin{pmatrix} A & B\\ -C & \tilde{A}\end{pmatrix} \tag{1}$$

where A, B and C are $n\times n$ matrices such that $\tilde{B}=-B$, $\tilde{C}=-C$ (the tilde denoting the transposed matrix). In the matrix elements A_{ij}, B_{ij}, C_{ij} and x_{kl} labels i and j run over $\beta=\{1,2,3\ldots n\}$ and labels k and l run over $\beta\cup\bar\beta$ with $\bar\beta=\{\bar 1,\bar 2,\bar 3\ldots\bar n\}$ and we have: $x_{ij}=A_{ij}$; $x_{i\bar j}=B_{ij}$; $x_{\bar i j}=-C_{ij}$; $x_{\bar i\bar j}=A_{ji}$. For $k=\bar i\in\bar\beta$ it is convenient to put $\bar k=i(\in\beta)$.

To matrix x (equation (1)) we associate a graph G consisting of: *n branches* denoted (i) or $(\bar i)$ with $i\in\beta$; branch $(i)=(\bar i)$ has two *extremities* i and $\bar i$ and is represented by an arrow going from i to $\bar i$.

For each pair of extremities k, l ($k\in\beta\cup\bar\beta$, $l\in\beta\cup\bar\beta$, $k\neq l$), a *passage* denoted $]k\ \ l[$ or $]l\ \ k[$ connecting these two extremities. Each passage is represented by an arrow and wears a *value* as shown in table 1. When the value of a passage is zero we can delete it from graph G. On figure 1 is represented the graph of the matrix x appearing for SU(3) in § 3 for which $A=0$. The three branches are represented by vertical segments.

The matrix x considered in H § 8 was also of form (1). In the case of a recoupling coefficient, all extremities were grouped in disjoint triplets. The extremities ijk of each

Table 1. Value and arrow of a passage.

Passage		Arrow from	to	Value
$]\bar i\ \ j[\,=\,]j\ \ \bar i[$ $\bar i\in\bar\beta,\ \ j\in\beta$		$\bar i$	j	A_{ij}
$]\bar i\ \ \bar j[\,=\,]\bar j\ \ \bar i[$ $\bar i\in\bar\beta,\ \ \bar j\in\bar\beta$	$i<j$	$\bar i$	$\bar j$	B_{ij}
$]i\ \ j[\,=\,]j\ \ i[$ $i\in\beta,\ \ j\in\beta$	$i<j$	i	j	C_{ij}

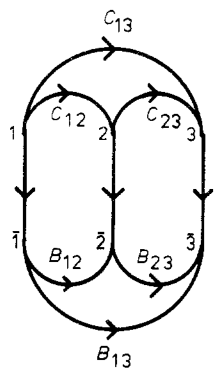

Figure 1. Graph of x for SU(3).

triplet were connected by three passages $]i\ j[$, $]j\ k[$ and $]k\ i[$ and no other passages existed. The graph defined in H was the same as here, to which was added a vertex for each above the triplet of extremities, thus forming the graph of the recoupling coefficient.

We define the various *diagrams* of G as in H § 4. We will not repeat these definitions here, but let us point out an imprecision in H § 4. In the sequences or cycles of alternating branches and passages connected each to the following one, it is also required that the sequence or cycle can be drawn on the graph without turning back at any point. For example $\ldots,]i\ j[, (j),]j\ k[, \ldots$ is not allowed since one has to turn back on branch (j) to draw the sequence.

However, we modify the definition of a free path: a free path is now a path beginning with a branch and ending with a passage.

We need to make the convention that there are $2n$ more free paths J_k $(k \in \beta \cup \bar{\beta})$ containing no passage and no branch. J_k can be thought of as containing only the extremity k. The definition of a simple diagram also has to be modified. The easiest way is to include in the set $\mathscr{B}(D)$ of the elements of D the extremities over which the diagram passes (each being counted as many times as the diagram passes over it). For example a simple diagram containing J_k will not pass over branch (k). We also define the set Ω_{kl} for $k \in \beta \cup \bar{\beta}$, $l \in \beta \cup \bar{\beta}$ formed of the open diagrams, having a free path of the form $(k),]\bar{k}\ p[, \ldots,]q\ l[$ (any p and q) or just $(k),]\bar{k}\ l[$ if $k \neq l$ or having the free path J_k if $k = l$.

The monomial $M(D)$ is defined for every diagram D in a way similar to H § 5. If D is a directed diagram, $M(D)$ is obtained by multiplying the following factors:

(a) for every passage belonging to $\mathscr{B}(D)$, the value of the passage (see table 1);

(b) for every element (branch or passage) of $\mathscr{B}(D)$ such that its direction in D is opposed to the arrow of that branch or passage in G, a factor -1;

(c) for every circuit in D, a factor -1.

Since $M(D)$ is independent of the directions of the circuits of D, $M(D)$ is also defined for non-directed diagrams. Note that $M(J_k) = 1$. We define for $k \in \beta \cup \bar{\beta}$, $l \in \beta \cup \bar{\beta}$ path P_{kl} made of branch (k) followed by passage $]\bar{k}\ l[$.

We have $x_{kl} = M(P_{kl})$ and the calculations of § 8.1 and the appendix of H remain valid†.

So we obtain

$$\det(1-x) = \Big(1 + \sum_{D \in K_G} M(D)\Big)^2 \tag{2}$$

where the sum is over the closed diagrams. For the example in figure 1, the closed diagrams and their $M(D)$ are given in figure 2 so that: $\det(1-x) = (1 - B_{12}C_{12} - B_{23}C_{23} - B_{13}C_{13})^2$. The graphical calculations in H § 8.2 give similarly $(1-x)^{-1}_{ij}$. However, there is a much simpler way of obtaining $(1-x)^{-1}_{ij}$ as we explain below.

Let X_{kl} be the cofactor of $(1-x)_{lk}$ in matrix $1-x$. For $i \in \beta$, $j \in \beta$ we have $X_{ij} = X_{\bar{j}\bar{i}}$, $X_{i\bar{j}} = -X_{j\bar{i}}$ and $X_{\bar{i}j} = -X_{\bar{j}i}$. It is then easily seen that the cofactors can be expressed in terms of the derivatives of $\det(1-x)$ with respect to A_{ij}, B_{ij} $(i<j)$ and C_{ij} $(i<j)$ for example:

$$(\partial/\partial A_{ij}) \det(1-x) = -X_{ji} - X_{\bar{i}\bar{j}} = -2X_{ji}.$$

† There is a misprint in the 6th line of the appendix: read $H'_G \cap \mathscr{R}_W$ instead of $H'_G \cup \mathscr{R}_W$.

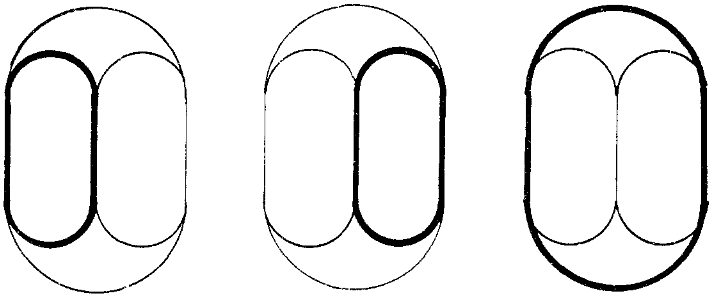

Figure 2. Closed diagrams of graph figure 1 and $M(D)$.

Calculating these derivatives from equation (2) we obtain

$$(1-x)^{-1}_{kl} = [\det(1-x)]^{-1}X_{kl} = \sum_{T\in\Omega_{kl}} M(T)\Big/\Big(1+\sum_{D\in K_G} M(D)\Big). \tag{3}$$

We will have to calculate

$$v(1-x)^{-1}w = \sum_{\substack{k\in\beta\cup\bar\beta\\ l\in\beta\cup\bar\beta}} v_k(1-x)^{-1}_{kl}w_l.$$

Putting $M'(T) = v_k M(T) w_l$ for $T\in\Omega_{kl}$ we have

$$v(1-x)^{-1}w = \sum_{T\in\Omega} M'(T)\Big/\Big(1+\sum_{D\in K_G} M(D)\Big). \tag{4}$$

For the example in figure 1,

$$(1-x)^{-1}_{11} = \frac{1-B_{23}C_{23}}{1-B_{12}C_{12}-B_{23}C_{23}-B_{13}C_{13}},$$

$$(1-x)^{-1}_{12} = \frac{B_{13}C_{23}}{1-B_{12}C_{12}-B_{23}C_{23}-B_{13}C_{13}}, \ldots$$

as can be seen from figure 3. If for this example $v_{\bar k} = w_{\bar k} = 0$ for $\bar k\in\bar\beta$ then

$$v(1-x)^{-1}w = \frac{vw-(v_1C_{23}+v_2C_{31}+v_3C_{12})(w_1B_{23}+w_2B_{31}+w_3B_{12})}{1-B_{12}C_{12}-B_{23}C_{23}-B_{13}C_{13}}.$$

3. Generating functions for $D^{[m]}_{MM'}(u)$

3.1. *Generating function for the basis vectors*

We now follow the notations of Henrich (1975). The polynomials $\Gamma(M;Z)$ in the elements of the $n\times n$ complex matrix Z, where

$$M = \begin{pmatrix} a_n & b_n & 0 & \ldots. & 0 \\ & a_{n-1} & b_{n-1} & 0 & .. & 0 \\ & & \ldots\ldots & & \\ & & a_2 & b_2 & \\ & & & a_1 & \end{pmatrix}$$

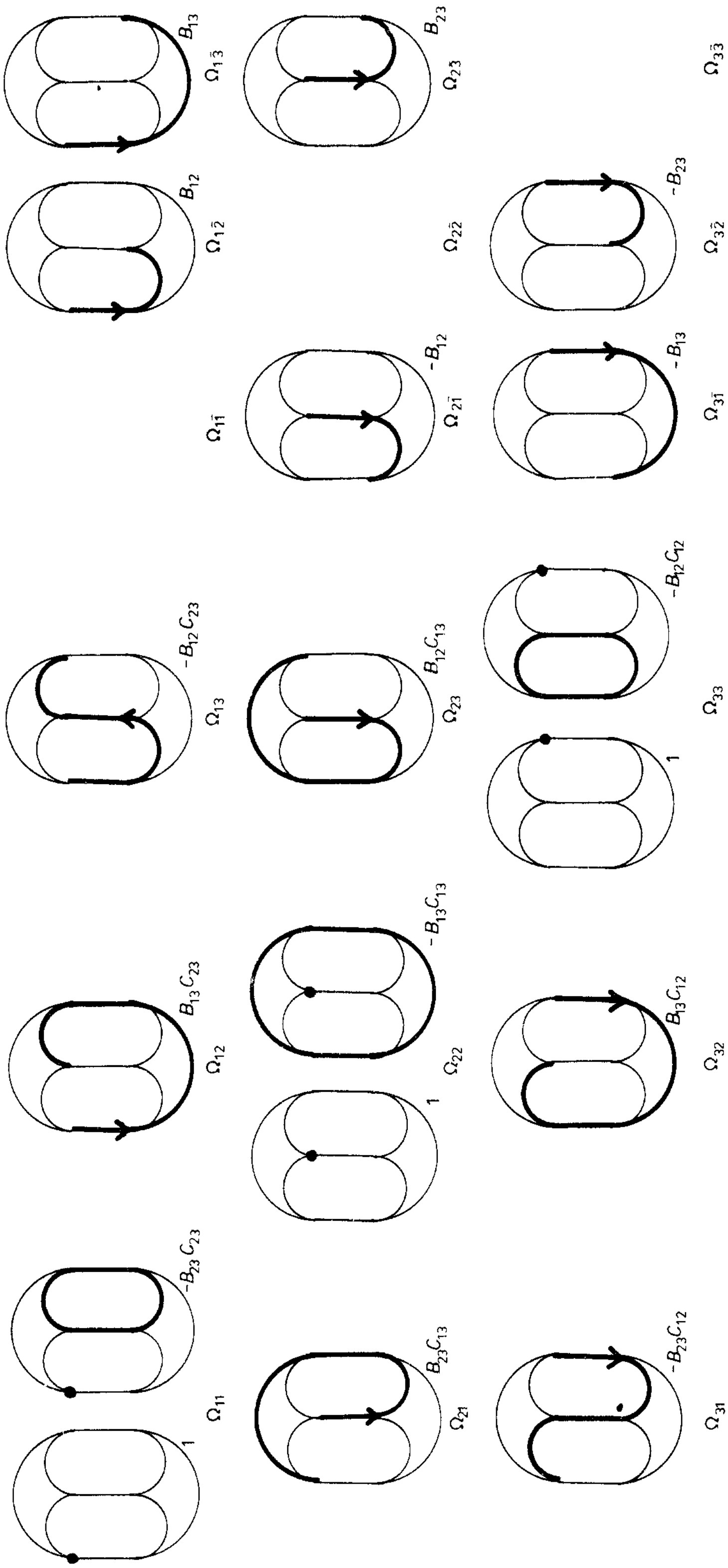

Figure 3. Some open diagrams of graph figure 1 and $M(D)$. J_1 is represented by a dot on extremity 1. All open diagrams from $\Omega_{k\bar{l}}$ ($k \in \beta$, any l) are represented.

is a Gel'fand pattern, form an orthonormal basis of the representation $[m]=[a_n\ b_n\ 0\ \dots\ 0]$ of SU(n), the Hermitian product being given by

$$(f, g)=\int \bar{f}(Z) g(Z)\, \mathrm{d}\gamma(Z)$$

where

$$\mathrm{d}\gamma(Z)=\pi^{-n^2} \exp(-\operatorname{Tr} Z^+ Z) \prod_{kl} \mathrm{d}x_{kl}\, \mathrm{d}y_{kl}, \qquad Z_{kl}=x_{kl}+\mathrm{i} y_{kl}.$$

$u \in \mathrm{SU}(n)$ operates on $\Gamma(M; Z)$ by $\Gamma(M; Z) \rightarrow \Gamma(M; Zu)$.

The generating function of the $\Gamma(M; Z)$ is given on page 2282 of Henrich in terms of $p=(p_1, p_2 \dots p_n)$, $q=(q_2, q_3 \dots q_n)$:

$$p_1=a_1, \qquad p_2=a_2-a_1, \dots, \qquad p_n=a_n-a_{n-1}$$

$$q_2=b_2, \qquad q_3=b_3-b_2, \dots, \qquad q_n=b_n-b_{n-1}$$

as

$$\sum_M N(M) \Gamma(M; Z) \lambda^p \mu^p=\exp\Big(\sum_j \lambda_j \Delta_j(u)+\sum_{j<k} \lambda_j \mu_k \Delta_{jk}(u)\Big) \tag{6}$$

where

$$\lambda^p=\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_n^{p_n}, \qquad \mu^q=\mu_2^{q_2} \mu_3^{q_3} \dots \mu_n^{q_n},$$

Σ_M means, as in the following, the sum over all patterns of type (5), $N(M)$ is a normalisation constant:

$$N(M)=\Big(\prod_{j=2}^n (a_j-b_{j-1}+1)!\Big)^{1/2}\Big\{(a_n-b_n)!\Big[\prod_{j=2}^n q_j! p_j! (a_{j-1}-b_j)! (a_j-b_j+1)\Big]\Big\}^{-1/2}$$

and where

$$\Delta_j(u)=u_{1j}, \qquad \Delta_{jk}(u)=\Delta_{jk}^{12}(u)$$

with

$$\Delta_{jk}^{lm}(u)=\det\begin{bmatrix} u_{lj} & u_{lk} \\ u_{mj} & u_{mk} \end{bmatrix}.$$

3.2. *Generating function for the representation matrix*

The representation matrix $\mathscr{D}_{MM'}^{[m]}(u)$ is defined by

$$\Gamma(M', Zu)=\sum_M \Gamma(M, Z) \mathscr{D}_{MM'}^{[m]}(u) \delta_{[M]_n,[M']_n}$$

$[M]_n$ denoting the top row of the Gel'fand pattern ($[m]=[M']_n$) or, using the orthogonality of the basis vectors, by:

$$\mathscr{D}_{MM'}^{[m]}(u)=\int \Gamma(M'; Zu) \bar{\Gamma}(M; Z)\, \mathrm{d}\gamma(Z).$$

A generating function for the $\mathscr{D}^{[m]}_{MM'}(u)$ is obtained from:

$$\Phi = \sum_{MM'} \mathscr{D}^{[m]}_{MM'}(u) N(M) N(M') \lambda^{p} \lambda^{q} \lambda'^{p'} \mu'^{q'}$$

$$= \int \exp\Big(\sum_k \lambda_k \overline{\Delta_k(Z)} + \sum_{k<l} \lambda_k \mu_l \overline{\Delta_{kl}(Z)} + \sum_j \lambda'_j \Delta_j(Zu) + \sum_{i<j} \lambda'_i \mu'_j \Delta_{ij}(Zu)\Big)\, \mathrm{d}\gamma(Z). \quad (7)$$

Putting $\zeta_k = Z_{1k}, \zeta_{\bar{k}} = \bar{Z}_{2k} \qquad (1 \leq k \leq n)$

$$\zeta = (\zeta_1 \zeta_2 \ldots \zeta_n \zeta_{\bar{1}} \zeta_{\bar{2}} \ldots \zeta_{\bar{n}}), \mathrm{d}\gamma(\zeta) = \pi^{-2n} \exp(-\bar{\zeta}\zeta) \prod_{i \in \beta \cup \bar{\beta}} (\mathrm{Re}\, \mathrm{d}\zeta_i)(\mathrm{Im}\, \mathrm{d}\zeta_i)$$

$$v_k = \sum_j u_{kj} \lambda'_j; \qquad v_{\bar{k}} = 0$$

$$w_k = \lambda_k; \qquad w_{\bar{k}} = 0$$

$$A_{kl} = 0$$

$$B_{kl} = \lambda_k \mu_l \qquad k<l \qquad \tilde{B} = -B$$

$$C_{kl} = \sum_{i<j} \lambda'_i \mu'_j \Delta^{kl}_{ij}(u)$$

the integral in equation (7) reduces to

$$\int \exp(\bar{\zeta} x \zeta + v\zeta + \bar{\zeta} w)\, \mathrm{d}\gamma(\zeta)$$

with x as in equation (1).

Its value, computed by the method of the appendix of Bargmann (1962), gives Φ as:

$$\Phi = [\det(1-x)]^{-1} \exp[v(1-x)^{-1} w]. \quad (8)$$

The various parts of equation (8) have been calculated in § 2: see equations (2) and (4); the example in figure 1, detailed in § 2, corresponds to the case of SU(3).

4. Conclusion

Due to equations (2) and (4), equation (8) for the generating function Φ has a compact form. However, the expressions for the representation matrices obtained from it involve a very great number of summations. Generalisation to other representations of SU(n) would require the generating functions of the basis vectors but these are unknown. By adding a term $\Sigma_{j<k<l} \lambda_j \mu_k \nu_l \Delta_{jkl}(u)$ in the exponent of equation (5) one does not obtain the generating function for the representations $[a_n\ b_n\ c_n\ 0\ \ldots\ 0]$ as can be seen in the case of SU(4) (Gazeau *et al* 1975 and 1978).

The proof of equation (2) is based on the expression of $\det(1-x)$ as an infinite product. The question remains of whether a more direct proof exists.

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